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Computation of Quasi-periodic Solutions of Forced Dissipative Systems

CHR. KAAS-PETERSEN

Laboratory of Applied Mathematical Physics, The Technical University of Denmark, DK-2800 Lyngby, Denmark

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The problem of finding a T-periodic solution of a forced dissipative system of ordinary differential equations is most conveniently reformulated as a fixed point problem of a Poincaré map mapping the phase space at $t=0$ into the phase space at $t=T$, and the stability of the periodic response is equivalent to the stability of the fixed point. It is shown how the problem of determining quasi-periodic solutions of forced systems with two forcing frequencies may be reformulated as a fixed point problem of a new type of Poincaré map that also opens the possibility of treating quasi-periodic solutions with more than two frequencies, as well as treating bifurcating quasi-periodic solutions of autonomous equations. Three examples are considered: one in which the exact solution is known, and two others where the quasi-periodic solutions have been determined by other means. © 1985 Academic Press, Inc.

1. INTRODUCTION

In this section we first state the problem. Second, we review a method used to solve this problem, and third we describe the new approach to the solution of the problem.

We consider a dissipative system of forced ordinary differential equations (ODE):

$$\dot{x} = f(t, x, \omega_1, \omega_2, \ldots, \omega_p)$$

$$(\dot{\ }) = \frac{d}{dt}(\ )$$

$$x \in R^n,\ f\colon R \times R^n \times R_+^p \to R^n \tag{1.1}$$

$$n \geqslant 1,\ p \geqslant 1$$

f is sufficiently smooth.

The p frequencies $\{\omega_1, \ldots, \omega_p\}$ are fixed. It is assumed that the steady state is a quasi-periodic function q where

$$q = q(\omega_1 t, \ldots, \omega_p t) \tag{1.2}$$

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and q is 2π-periodic in each of its arguments. If q shall be a true quasi-periodic function the numbers $\omega_1, ..., \omega_p$ must be incommensurable, which means that no vanishing linear combination $c_1\omega_1 + \cdots + c_p\omega_p$ exists with rational coefficients $c_1, ..., c_p$.

The number of frequencies can be reduced if $\{\omega_1, ..., \omega_p\}$ is not incommensurable. In our algorithm, however, we permit commensurable as well as incommensurable frequencies. For $p=1$ we have the problem of determining a periodic solution of (1.1). Here shooting methods are efficient, which try to find a solution that take on the same value in both ends of the interval $[0, T]$, where $T = 2\pi/\omega_1$.

Let us reformulate this by introducing the Poincaré map. The Poincaré map has domain in state space at $t=0$ and range in state space at $t=T$. Let $x(t)$ be the solution of (1.1) for which $x(0)=x_0$. Then the Poincaré map P maps x_0 into $x(T)$, that is, $x(T)=P(x_0)$. The periodic solution $q(t)$ is then a fixed point for the map P, that is, $q(T)=q(0)$, and the stability of the periodic solution is equivalent to the stability of the fixed point. For $p=2$ we seek a quasi-periodic solution, $q(\omega_1 t, \omega_2 t)$. The only method known to the author is due to Chua and Ushida [1]. They assume a generalized Fourier series

$$
\begin{aligned}
&x = a_0 + \sum_{i=1}^{\infty} [a_{2i-1} \cos v_i t + a_{2i} \sin v_i t] \\
&v_i = m_{1i}\omega_1 + \cdots + m_{pi}\omega_p \qquad (1.3) \\
&m_{1i}, ..., m_{pi} \text{ are integers such that } v_i > 0.
\end{aligned}
$$

They truncate the series at a certain number M, and derive a system of equations for the coefficients (vectors) $a_0, a_1, a_2, ...$.

We shall instead use a Poincaré map. First we have to define a stroboscopic function, s. If $x(t)$ is the solution of (1.1) with $x(0)=x_0$, we let the stroboscopic function take on the values

$$x(0), x(T_1), x(2T_1), x(3T_1), ..., T_1 = \frac{2\pi}{\omega_1}$$

in the points $\tau_0, \tau_1, \tau_2, \tau_3, ...$ where

$$\tau_k = kT_1 \text{ modulo } T_2, \qquad k = 0, 1, 2, ..., T_2 = \frac{2\pi}{\omega_2}.$$

We so to speak eliminate one of the 2π-periodic arguments of q.

The Poincaré map P then maps $s(0)$ ($= x(0)$) into $s(T_2)$, thus $s(T_2) = P(s(0))$. We shall in fact use a slightly different definition of τ_k which allow us to compute $s(T_2)$ by interpolation. We have found the quasi-periodic solution when

$$s(T_2) = s(0).$$

When three periods T_1, T_2, and T_3 are given, we may define a stroboscopic function s of two variables. The first variable is $\tau^{[1]} = t \bmod T_2$ and the second is $\tau^{[2]} = t \bmod T_3$. The Poincaré map then maps $s(0, 0) = x(0)$ into $s(T_2, T_3)$. We have found the quasi-periodic solution when $s(T_2, T_3) = s(0, 0)$.

2. Method

In this section we first consider the Poincaré map when one frequency ω is given and show how to use this map in obtaining periodic solutions. Next we describe the new Poincaré map which makes it possible to determine the quasi-periodic solution. In both cases the Poincaré map transforms the problem of finding periodic and quasi-periodic solutions to that of finding fixed points. In the next section we describe how to determine the stability of the fixed point. Poincaré maps for autonomous equations are well known; they are described in [4].

We first describe the Poincaré map when $p = 1$ by considering the equation

$$\dot{x} = f(t, x, \omega), \qquad x \in R^n \tag{2.1}$$

where f is periodic with period $T = 2\pi/\omega$, thus $f(t, x, \omega) = f(t + T, x, \omega)$. We want to find the T-periodic solution $q(t)$ of (2.1) for which

$$\begin{aligned} \dot{q} &= f(t, q, \omega) \\ q(0) &= q(T). \end{aligned} \tag{2.2}$$

Now let $u(0)$ be an approximation to $q(0)$.

We define the Poincaré map P as follows:

$$\begin{aligned} P&: R^n \to R^n \\ u(T) &= P(u(0)) \end{aligned} \tag{2.3}$$

where $u(t)$ is the solution of (2.1) with $u(0)$ as the initial value. What does P do? It maps a point $u(0)$ in state space at $t = 0$ into that point $u(T)$ in state space at $t = T$ where the trajectory through $u(0)$ ends at $t = T$. This map may not be defined in all of the state space, but it is certainly defined in some neighbourhood of the periodic solution. The periodic solution is seen to be a fixed point of P.

The map P can be used to define a map Q as follows:

$$Q = P - I \tag{2.4}$$

where I is the identity. Of course the domain and the range of Q equal the domain and the range of P.

We see that Q maps the periodic solution into a zero. Newton's method is very useful for the purpose of finding zeros. The values of $Q(x)$ and the derivative $DQ(x)$

of $Q(x)$ needed in Newton's method are computed numerically. As an example we find the jth column of $DQ(x)$ when

$$y = Q(x).$$

We perturb x by the vector δ_j in the jth coordinate direction, compute

$$y_j = Q(x + \delta_j)$$

and use the difference approximation:

$$j\text{th column of } DQ = \frac{y_j - y}{\|\delta_j\|}.$$

When all columns of $DQ(x)$ has been computed the Newton method yields

$$x_{\text{new}} = x - [DQ(x)]^{-1} y, \qquad y = Q(x) \tag{2.5}$$

and x_{new} shall be used in the next iteration.

This process is repeated until $\|Q(x)\| < \varepsilon$ for some preassigned ε. The process is stopped if too many iterations are performed. This could be an indication of a too poor initial guess.

Now let us take $p = 2$, and we have the quasi-periodic solution $q(\omega_1 t, \omega_2 t)$ of

$$\dot{x} = f(t, x, \omega_1, \omega_2) \tag{2.6}$$

where q is 2π-periodic in both arguments. Take $t_k = kT_1 = k \cdot 2\pi/\omega_1$, $k \in Z_+$. Define

$$\begin{aligned} \tau_k &= 1 + \hat{t}_k \qquad \text{if} \quad 0 \leqslant \hat{t}_k < \tfrac{1}{2} \\ &= \hat{t}_k \qquad\quad\ \ \text{if} \quad \tfrac{1}{2} \leqslant \hat{t}_k < 1 \end{aligned} \tag{2.7}$$

where $\hat{t}_k = (t_k/T_2) \bmod 1$.

Then we may define a (vector) function $s(\cdot)$, s: $\{0\} \cup [\frac{1}{2}, \frac{3}{2}] \to R^n$, as follows:

$$\begin{aligned} s(0) &= q(0, 0) \\ s(\tau_k) &= q(\omega_1 t_k, \omega_2 t_k) = q\left(\omega_1 \cdot k \frac{2\pi}{\omega_1}, \omega_2 t_k\right) \\ &= q\left(0, 2\pi \frac{t_k}{T_2}\right) = q(0, 2\pi \hat{t}_k) = q(0, 2\pi\tau_k) \qquad \text{for} \quad k \geqslant 1. \end{aligned} \tag{2.8}$$

We shall call s the stroboscopic function. Equations (2.8) show that if q is quasi-periodic, then s is periodic with period 1 on any interval of length 1, so $s(0) = s(1)$. However, no k can be chosen to make $\tau_k = 1$, but let K be a finite set of k's ordered increasingly such that

$$1 - \varepsilon_q \leqslant \tau_k \leqslant 1 + \varepsilon_q, \qquad \varepsilon_q > 0, \quad k \in K \tag{2.9}$$

and then interpolate each coordinate of s on the net τ_k, $k \in K$, to obtain $s(1)$.

Let $u(0)$ be some approximation to $q(0, 0)$, and let $u(t)$ be the solution of (2.6) with $u(0)$ as initial value. Then we set

$$s(\tau_k) = u(kT_1) \qquad \text{for} \quad k \in K; \qquad s(0) = u(0). \tag{2.10}$$

We interpolate the stroboscopic function to find $s(1)$. Therefore we define the Poincaré map by

$$\begin{aligned} &P: R^n \to R^n \\ &s(1) = P(s(0)) = P(u(0)). \end{aligned} \tag{2.11}$$

It should be added that s is not periodic, not even continuous, unless $u(0)$ is on the quasi-periodic solution. s can be made continuous if we restrict k to a subset $\tilde{K}$ of K, where τ_k is a monotonic increasing or decreasing sequence for $k \in \tilde{K}$.

The problem of finding the quasi-periodic solution is hereby reduced to the problem of finding the fixed point of (2.11).

We shall also indicate how the case $p = 3$ is treated in a similar way. This may easily be extended to higher values of p. We seek the solution $q(\omega_1 t, \omega_2 t, \omega_3 t)$ of

$$\dot{x} = f(t, x, \omega_1, \omega_2, \omega_3). \tag{2.12}$$

q is 2π-periodic in each argument. Take $t_k = k \cdot T_1$, $k \in Z_+$. Define

$$\begin{aligned} \tau_k^{[l]} &= 1 + \hat{t}_k^{[l]} \qquad \text{if} \quad 0 \leqslant \hat{t}_k^{[l]} < \tfrac{1}{2}, \\ &= \hat{t}_k^{[l]} \qquad\quad\ \ \text{if} \quad \tfrac{1}{2} \leqslant \hat{t}_k^{[l]} < 1, \end{aligned} \qquad l = 1, 2 \tag{2.13}$$

where

$$\hat{t}_k^{[1]} = \frac{t_k}{T_2} \bmod 1$$

$$\hat{t}_k^{[2]} = \frac{t_k}{T_3} \bmod 1.$$

Equation (2.13) implies that the points $(\tau_k^{[1]}, \tau_k^{[2]})$, $k \in Z_+$, are inside a square in R^2 with center in $(1, 1)$ and side 1. When $t = t_k$, $k \in Z_+$, we consider q on this square, because the 2π-periodicity in each argument of q yields

$$q(\omega_1 t_k, \omega_2 t_k, \omega_3 t_k) = q\left(0, \frac{2\pi}{T_2} t_k, \frac{2\pi}{T_3} t_k\right) = q(0, 2\pi\tau_k^{[1]}, 2\pi\tau_k^{[2]}).$$

We define the stroboscopic (vector) function $s(\cdot, \cdot)$, s: $\{(0, 0)\} \cup [\tfrac{1}{2}, \tfrac{3}{2}] \times [\tfrac{1}{2}, \tfrac{3}{2}] \to R^n$, in the following way:

$$\begin{aligned} s(0, 0) &= q(0, 0, 0) \\ s(\tau_k^{[1]}, \tau_k^{[2]}) &= q(0, 2\pi\tau_k^{[1]}, 2\pi\tau_k^{[2]}) \qquad \text{for} \quad k \geqslant 1. \end{aligned} \tag{2.14}$$

We see that s is periodic in each argument with period 1, so $s(0, 0) = s(1, 1)$.

Let K be a finite subset of k-values ordered increasingly such that $(\tau_k^{[1]}, \tau_k^{[2]})$ is inside a small square with center in $(1, 1)$ and side $2\varepsilon_q$. Thus

$$1-\varepsilon_q \leqslant \tau_k^{[l]} \leqslant 1+\varepsilon_q, \qquad l=1, 2, \text{for } k \in K. \tag{2.15}$$

Let $u(0)$ be some approximation to $q(0, 0, 0)$ and $u(t)$ the solution of (2.12) with $u(0)$ as initial value. Then we set

$$s(\tau_k^{[1]}, \tau_k^{[2]}) = u(kT_1) \qquad \text{for} \quad k \in K; s(0, 0) = u(0). \tag{2.16}$$

Each coordinate of the stroboscopic function is interpolated on the net $(\tau_k^{[1]}, \tau_k^{[2]})$, $k \in K$, to find $s(1, 1)$. The Poincaré map is now defined by:

$$\begin{aligned} &P: R^n \to R^n \\ &s(1, 1) = P(s(0, 0)) = P(u(0)). \end{aligned} \tag{2.17}$$

The quasi-periodic problem is hereby again reduced to a fixed point problem.

3. Stability of the Fixed Point

Stability of periodic solutions of forced ODE with one forcing frequency is usually determined from the Floquet multipliers [2], but could equally well have been determined as the stability of the fixed point. Here we shall use the latter approach, since this also can be used in the case of quasi-periodic functions.

Let x be the fixed point of the map P, so

$$x = P(x). \tag{3.1}$$

If ε is any disturbance, then by Taylor's theorem (since f is sufficiently smooth)

$$P(x+\varepsilon) = P(x) + DP(x)\,\varepsilon + O(\|\varepsilon\|^2). \tag{3.2}$$

Let $x+\varepsilon$ be mapped into $x+\delta$, then

$$x+\delta = P(x+\varepsilon) = P(x) + DP(x)\,\varepsilon + O(\|\varepsilon\|^2). \tag{3.3}$$

Retaining only the lowest-order terms we get

$$\delta = DP(x)\,\varepsilon. \tag{3.4}$$

x is stable when any disturbance ε yields a δ such that $\|\delta\| < \|\varepsilon\|$. This is fulfilled if all eigenvalues of $DP(x)$ are inside the unit circle.

Another formulation is that P should be a contraction in a neighbourhood of x, and we can get the initial point by merely iterating the map P, starting with some x_0 near to x. This we shall call the brute force method.

Note that since $Q = P - I$ we get

$$DP = DQ + I. \tag{3.5}$$

4. Comments and Practicalities

In this section we will comment on the following issues:

(a) choice of initial conditions
(b) choice of interpolating functions
(c) choice of delta in computation of DQ
(d) applied software
(e) relation to bifurcation problems.

(a) *Choice of Initial Conditions*

Some good guess on the initial conditions must be known. This requirement arises from the nature of the Newton method. For linear ODEs the Newton method will determine the solution in one iteration, and therefore we may use any initial values, as long as the solution of the ODE can be determined numerically.

(b) *Choice of Interpolating Functions*

Here we shall make some comments on the interpolation in the case $p=2$. By construction of the Poincaré map, we have to interpolate the stroboscopic function s in some interval around 1.

The τ_k's cannot be chosen at will and the derivative of s is not available.

On the other hand we can take a smaller ε_q or include more points. To obtain a reasonable interpolation we have used natural cubic splines which are preferable when s is not continuous, and some jumps in the values of s may be observed.

(c) *Choice of Delta*

The delta is used in numerical approximation of the derivative. Let $g: R^n \to R$ be one of the functions in the map Q. Then the derivative in the direction of v, $\|v\| = 1$, is

$$\frac{\partial g}{\partial v} = \frac{g(x+\delta v) - g(x)}{\delta} + O(\delta) \tag{4.1}$$

and the error is therefore proportional to δ. The statement follows from the Taylor series expansion of g. So δ must be small in some sense. But g is computed with some error, say, proportional to ε_g and the numerator of (4.1) is therefore in error proportional to ε_g, so that

$$\text{error}\left(\frac{g(x+\delta v) - g(x)}{\delta}\right) = O\left(\frac{\varepsilon_g}{\delta}\right)$$

according to which we shall choose δ large. The optimal δ should minimize the total error in $\partial g/\partial v$, that is, minimize

$$E(\delta) = \frac{\varepsilon_g}{\delta} + \delta.$$

The optimal δ is $\delta = \varepsilon_g^{1/2}$ and the error in $\partial g/\partial v$ is proportional to $\varepsilon_g^{1/2}$. We must therefore make an estimate of ε_g. First let us focus on the interpolation error. Let n_i be the number of knots in an interval of length $2 \cdot \varepsilon_q \cdot T_1$. The knots are randomly distributed. Let h be the maximal distance between two neighbouring knots. Then Kershaw [3] has shown that the maximal approximation error

$$e(\tau) \leqslant Kh^4$$

for τ in the subinterval

$$\tau_a \leqslant \tau \leqslant \tau_b$$

where $\tau_a - \min_{k \in K}\{\tau_k\} = O(h \ln h)$ and $\max_{k \in K}\{\tau_k\} - \tau_b = O(h \ln h)$ for $h \to 0$. As an approximation of h, place the n_i knots equidistant in the interval, so

$$h = \frac{2 \cdot \varepsilon_q \cdot T_1}{n_i - 1}. \tag{4.2}$$

Second, an error arises from the numerical solution of the ODE. We have not tried to estimate it. But the error will be significant when n_i is large. So when n_i is small the interpolation is error-determining, and when n_i is large the numerical solution of the ODEs is error-determining. For n_i small, ε_g is proportional to h^4 and a proper δ may be taken to be proportional to h^2, where h is given by (4.2). We have used $\delta = 10^{-4}$ based on some experimentation.

(d) *Applied Software*

All calculations were performed in double precision on an IBM 3033 machine. Solution of the ODE was performed by the IMSL-routine DVERK using 5th- and 6th-order Runge–Kutta methods with variable steplength, which is efficient when the equations are non-stiff. The local tolerance was set to 10^{-8}. The interpolation by splines was performed by the IMSL-routines ICSCCU and ICSEVU.

(e) *Relation to Bifurcation Theory*

Equation (1.1) for which we want the quasi-periodic solution can be generalized by introducing some parameter v. Thus we consider

$$\dot{x} = f(t, x, \omega_1, \ldots, \omega_p, v). \tag{4.3}$$

Then the quasi-periodic solution is also a function of v, and we can "follow" the solution as a function of v. Bifurcation then takes place when the stable solution becomes unstable as a result of one or more of the eigenvalues of DP crossing the unit circle.

In the above formulation the quasi-periodic solution was present from the very beginning. We will call it the basic solution. But quasi-periodic solutions may also arise as a result of bifurcation, for instance, in autonomous equations, where a quasi-periodic solution may bifurcate from a periodic solution. Also in a forced system with one period T, the basic T-periodic solution may bifurcate into a quasi-periodic solution.

In order to investigate bifurcation it is necessary to determine the eigenvalues of DP. The method described in Section 2 can be modified to handle the case of quasi-periodic solutions in autonomous ODEs. This will be the subject of a forthcoming paper. A problem arises due to the fact that the two periods T_1 and T_2 are not explicitly given in the equation. All that is known (or can be computed) is the return time for the Poincaré map. The return time will in general depend on the initial condition.

EXAMPLE 1. We consider the linear differential equation

$$\ddot{x} + 2\alpha\dot{x} + \beta x = (\beta - 2)\cos\sqrt{2}\,t - 2\sqrt{2}\,\alpha\sin\sqrt{2}\,t + (\beta - 1)\cos t - 2\alpha\sin t$$

where α, β are real, $\alpha^2 > \beta$. Thus $\omega_1 = \sqrt{2}$ and $\omega_2 = 1$, so $T_1 = \sqrt{2}\,\pi$ and $T_2 = 2\pi$. The complete solution is

$$x = c_1 e^{r_+ t} + c_2 e^{r_- t} + \cos\sqrt{2}\,t + \cos t$$

where $r_\pm = -\alpha \pm \sqrt{\alpha^2 - \beta}$, $\alpha^2 - \beta > 0$. c_1, c_2 are real constants. The transient part, which is the solution of the homogeneous equation, is

$$x_{\text{trans}} = c_1 e^{r_+ t} + c_2 e^{r_- t}$$

and the steady-state part, which is the particular solution, is

$$q(\omega_1 t, \omega_2 t) = \cos\sqrt{2}\,t + \cos t.$$

Since this equation is linear, and the right-hand side consists of a T_1-periodic part added to a T_2-periodic part, it could have been solved using the superposition principle.

We use $n_i = 4$ points in the interpolation and $\varepsilon_q = 0.2$ and find that $t_k = kT_1$ for $k = 4, 7, 10, 13$ will give an increasing sequence of τ_k.

For $\alpha = 2$, $\beta = 1$ we have $r_\pm < 0$ and with $(x, \dot{x}) = (0, 0)$ as initial condition we obtain the solution in one Newton iteration

$$x = 1.992720023$$

$$\dot{x} = -0.0003792152.$$

It compares favourably with the exact solution $(x, \dot{x}) = (2, 0)$. The eigenvalues of the Poincaré map are

$$\lambda_1 = -0.00038726$$

$$\lambda_2 = 0 \qquad \text{(within computing accuracy)}$$

thus confirming the stability of the solution.

For $t_k = kT_1 = k\sqrt{2}\,\pi$, $k = 0, 1, \ldots$, the stroboscopic function is

$$s(\hat{t}_k) = \begin{bmatrix} x(\hat{t}_k) \\ \dot{x}(\hat{t}_k) \end{bmatrix} = \begin{bmatrix} 1 + \cos(2\pi\hat{t}_k) \\ -\sin(2\pi\hat{t}_k) \end{bmatrix}$$

where

$$\hat{t}_k = \frac{t_k}{T_2} \bmod 1.$$

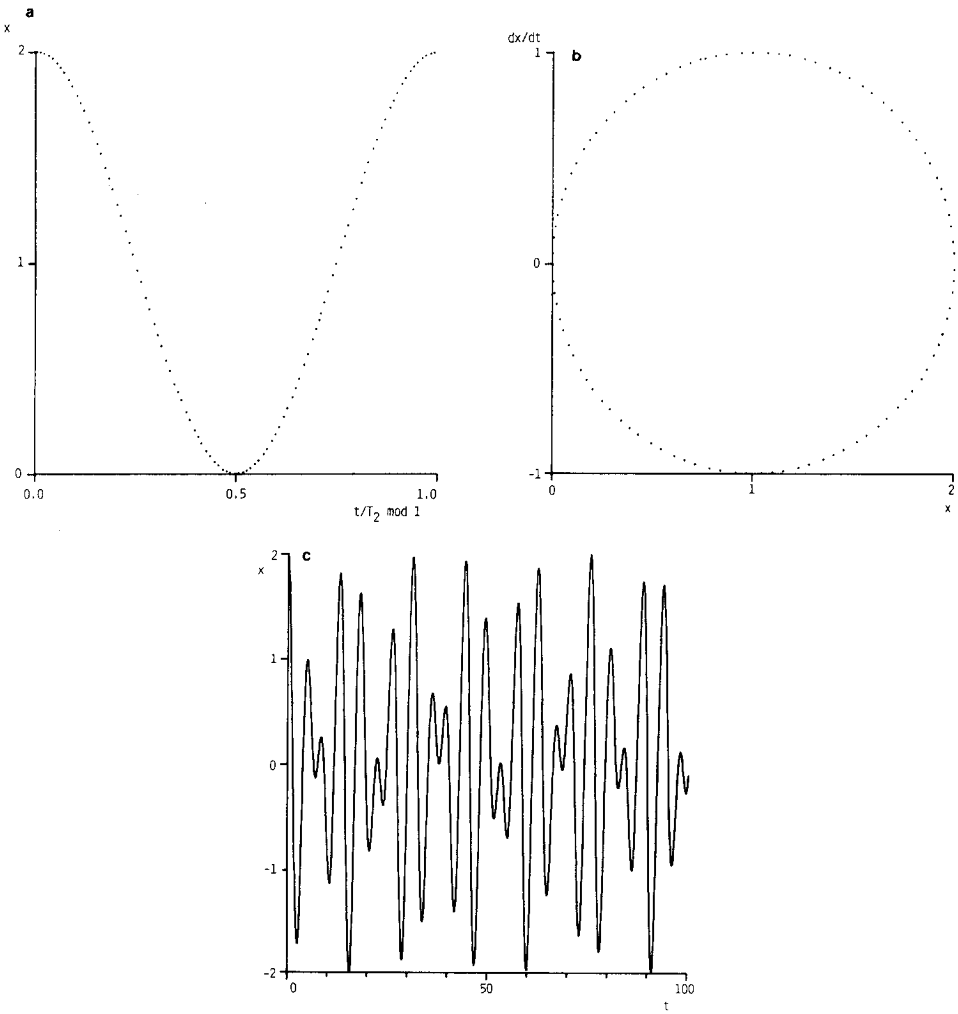

FIG. 1. Quasi-periodic steady-state response of the linear equation $\ddot{x} + 4\dot{x} + x = -\cos\sqrt{2}\,t - 4\sqrt{2}\sin\sqrt{2}\,t - 4\sin t$. Initial conditions $(x, \dot{x}) = (1.99272002, -0.0003792)$ at $t = 0$. (a) First component of the stroboscopic function $s(\hat{t}_k)$ for $k = 0, \ldots, 100$. (b) Second component of $s(\hat{t}_k)$ versus first component of $s(\hat{t}_k)$ for $k = 0, \ldots, 100$. This we call the strobed trajectory. (c) Time evolution of x, $0 \leqslant t \leqslant 100$.

In Fig. 1a we have plotted the first component of s versus $\hat{t}_k$, and in Fig. 1b we have shown $\dot{x}(\hat{t}_k)$ as a function of $x(\hat{t}_k)$. This is denoted the strobed trajectory. A plot of x as a function of time t is shown in Fig. 1c for $0 \leqslant t \leqslant 100$. For $\alpha = 1/20$ and $\beta = -1/100$ we obtain $r_{\pm} = (1 \pm \sqrt{5})/20$ and $r_{+} > 0$ so we have an unstable solution. In one Newton iteration we get

$$x = 1.963136492$$

$$\dot{x} = 0.000305991$$

and the eigenvalues

$$\lambda_1 = 0.73403979$$

$$\lambda_2 = 61.71872713$$

thus confirming the instability of the solution.

EXAMPLE 2. This example is discussed in [1]. We consider the Duffing equation

$$\ddot{x} + 0.05\dot{x} + x + x^3 = 0.3 \cos t + 1.5 \cos 0.115t.$$

We see that $\omega_1 = 1.0$ and $\omega_2 = 0.115$, so that the forcing function is not a true quasi-periodic solution, since it is periodic with period $T = 200 \cdot 2\pi$. However, T is too large for use of the Poincaré map for the periodic case. Using (0, 0) as initial value the brute force method converged to the fixed point in 5 iterations using a 7-knot spline interpolation, where $K = \{8, 9, 17, 18, 26, 34, 35\}$ (so $\varepsilon_q = 0.1$).

$$(x, \dot{x}) = (1.219273582, 0.3004775330) \tag{5.1}$$

and the eigenvalues are

$$\lambda_{1,2} = -0.02604 \pm i\,0.05943$$

using (5.1) as initial values.

(0, 0) is not close to the fixed point, and therefore Newton's method used 4 iterations, and determined

$$(x, \dot{x}) = (1.219294186, 0.3005114519). \tag{5.2}$$

Using (5.2) as initial values the eigenvalues are

$$\lambda_{1,2} = -0.02894 \pm i\,0.05801.$$

In both cases $|\lambda_{1,2}| < 1$, so the solution is stable. Otherwise the brute force method would not converge.

The stroboscopic function, the strobed trajectory and the evolutional behaviour of x are plotted in Figs. 2a,b,c using (5.2) as initial values.

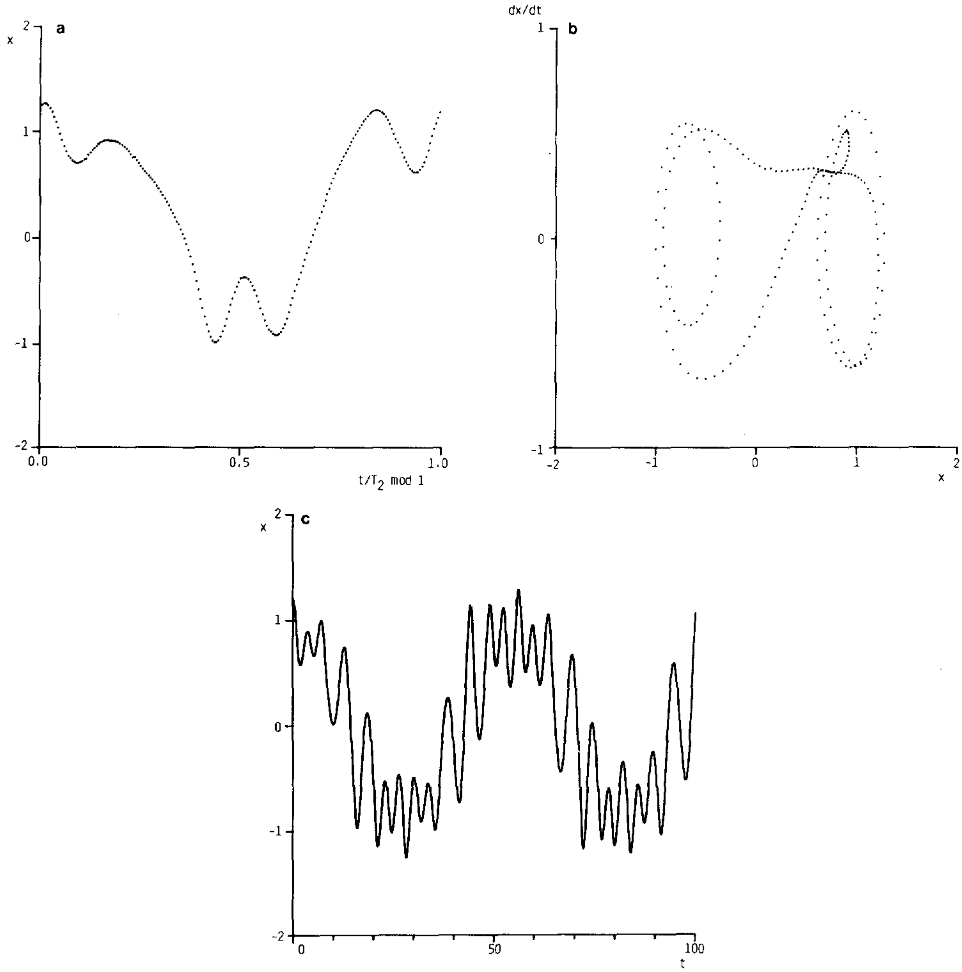

FIG. 2. Quasi-periodic steady-state response of Duffing's equation $\ddot{x}+0.05\dot{x}+x+x^3=0.3\cos t+1.5\cos 0.115t$. Initial conditions $(x,\dot{x})=(1.219294, 0.30051145)$ at $t=0$. (a) First component of stroboscopic function, i.e., $x(kT_1)$, $k=0, 1,..., 200$, $T_1=2\pi$, versus the independent variable, $\hat{t}_k$. (b) The strobed trajectory, $k=0, 1,..., 200$. (c) Time evolution of x, $0\leqslant t\leqslant 100$.

In [1] Chua and Ushida obtained the initial values

$$(x,\dot{x})=(1.21332, 0.33872)$$

which compares very well with our results.

EXAMPLE 3. This example is discussed in [1]; we have another Duffing equation

$$\ddot{x}+0.1\dot{x}+x+x^3=(1+\cos 0.115t)\cos t.$$

T_1 and T_2 are as in Example 2.

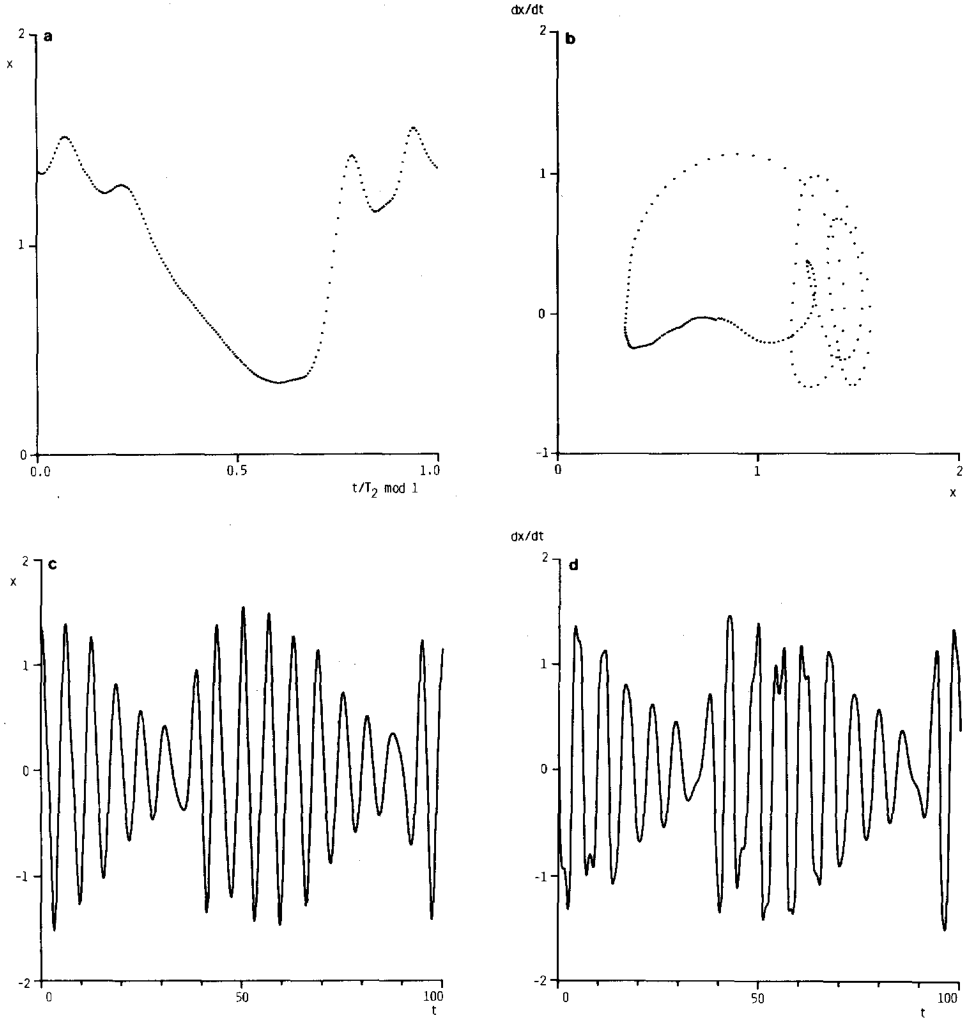

FIG. 3. Quasi-periodic steady-state response of Duffing's equation $\ddot{x}+0.1\dot{x}+x+x^3=(1+\cos 0.115t)\cos t$. Initial conditions $(x, \dot{x})=(1.354439, 0.1390661)$ at $t=0$. (a) First component of stroboscopic function, $k=0, 1, \ldots, 200$. (b) The strobed trajectory, $k=0, 1, \ldots, 200$. (c) Time evolution of x, $0 \leqslant t \leqslant 100$. (d) Time evolution of $\dot{x}$, $0 \leqslant t \leqslant 100$.

Using $\tilde{K}=\{8, 17, 26, 35\}$ (so $\varepsilon_q=0.2$) and taking $(x, \dot{x})=(0, 0)$ as initial point, Newton's method reaches the initial values as the steady-state solution in two iterations

$$(x, \dot{x})=(1.3494521, 0.12085215).$$

The eigenvalues of the Poincaré map were found to be

$$\lambda_{1,2}=0.0018798 \pm i\,0.0012263.$$

If we instead use $K = \{1, 7, 8, 9, 10, 16, 17, 18, 19, 25, 26, 27, 34, 35, 36\}$, $\varepsilon_q = 0.2$, we obtain

$$(x, \dot{x}) = (1.354439, 0.1390661)$$

in two Newton iterations.

The plots in Fig. 3 have been produced with this initial condition.

In [1] Chua and Ushida found

$$(x, \dot{x}) = (1.35403, 0.15168)$$

which compares favourably with our result.

Conclusions

The Poincaré map, which has already proved useful for the determination of periodic solutions of autonomous ODEs, is also applicable to the determination of periodic solutions of forced ODEs with one forcing frequency. Its applicability is due to the fact that only one point on the trajectory is considered. In this way the periodic solution becomes a fixed point in phase space, and the stability of a fixed point is equivalent to the stability of the periodic solution. Furthermore the method permits us to calculate stable and unstable solutions with equal ease. In this article these merits are extended to the case of quasi-periodic solutions.

Generalized Fourier series can be used to find quasi-periodic solutions as well as Fourier series can be used to find periodic solutions. In neither case, however, can they be used to determine the stability of the solutions.

In addition the Poincaré map yields a clear picture of important qualitative aspects of the quasi-periodic solutions.

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